

THE SOCLE SERIES OF A LEAVITT PATH ALGEBRA

GENE ABRAMS, KULUMANI M. RANGASWAMY, AND MERCEDES SILES MOLINA

ABSTRACT. We investigate the ascending Loewy socle series of Leavitt path algebras $L_K(E)$ for an arbitrary graph E and field K . We classify those graphs E for which $L_K(E) = S_\lambda$ for some element S_λ of the Loewy socle series. We then show that for any ordinal λ there exists a graph E so that the Loewy length of $L_K(E)$ is λ . Moreover, $\lambda \leq \omega$ (the first infinite ordinal) if E is a row-finite graph.

Graph C^* -algebras comprise an important class of C^* -algebras. These have been extensively studied over the past ten years (see e.g. [21]), and include as specific examples the Cuntz-Krieger algebras, the Toeplitz algebra, and many of the AF-algebras. Recently the investigation of the algebraic aspects of the analogues of these C^* -algebras, the *Leavitt path algebras* $L_K(E)$, have become an active topic of research (see e.g. [1], [2], [3], [9], [10], [11], and [12], among others). This note, in which we study the ascending Loewy socle series of $L_K(E)$ (see Definition 1.1), is one more contribution in the study of the algebraic structure of Leavitt path algebras over an arbitrary graph E and arbitrary field K . Unlike many of the earlier articles written on the topic of Leavitt path algebras, here an *arbitrary* graph E means there are neither cardinality restrictions nor graph-theoretical restrictions (e.g. row-finiteness) imposed on E .

W. Krull ([19], [20]) was the first to develop the interrelationships between the socle series of a ring and its ideal structure; this was achieved in the context of Noetherian rings. Subsequently, in the case of rings satisfying minimum conditions modulo the Jacobson radical, the socle series turned out to be quite useful (as shown, for example, by Artin-Nesbitt-Thrall [13] and Dauns [16]).

Our goal in this article is to describe the socle series of a Leavitt path algebra $L_K(E)$ in terms of the graph E . Specifically, we achieve two main results: we give in Theorem 3.2 an explicit description of those graphs E for which the socle series terminates at $L_K(E)$ (following [15], we call such a ring a *Loewy ring*); and in Theorem 4.1 we construct, for each ordinal λ , a graph E for which $L_K(E)$ is a Loewy ring of length λ . In contrast, we show in Theorem 4.2 that if E is a row-finite graph then the Loewy length of $L_K(E)$ must be finite or ω (the first countable ordinal). Further, we show that if $L_K(E)$ is a Loewy ring then necessarily E is an acyclic graph, but not conversely. In particular, we conclude that, for any graph E , if $L_K(E)$ is a Loewy ring then $L_K(E)$ must be von Neumann regular.

2000 *Mathematics Subject Classification.* Primary: 16S99, 16D60, 46L89; Secondary: 16W50.

Key words and phrases. Leavitt path algebra, ascending Loewy socle series.

The first author is partially supported by the U.S. National Security Agency under grant number H89230-09-1-0066. The third author is partially supported by the Spanish MEC and Fondos FEDER through project MTM2007-60333, and jointly by the Junta de Andalucía and Fondos FEDER through projects FQM-336 and FQM-2467. This work was initiated while the third author was hosted as a Research Visitor by the Department of Mathematics at the University of Colorado at Colorado Springs. She thanks this host center for its warm hospitality and support.

1. BASIC CONCEPTS: LOEWY RINGS, AND THE SOCLE OF $L_K(E)$

In this first section we describe the ascending Loewy socle series of an arbitrary ring, and then study this series in the context of Leavitt path algebras. Along the way we provide a description of the socle of $L_K(E)$ for an arbitrary graph E .

The Loewy series of a module M over a ring R is defined in terms of its socle. Recall that the *socle* $Soc(M)$ of an R -module M is the sum of all its simple submodules (where $Soc(M)$ is defined to be $\{0\}$ in case M has no simple submodules). In particular, if we consider R as a left R -module, then the *left socle* $Soc_l(R)$ is the sum of all the simple left ideals of R . Similarly, one defines the *right socle* $Soc_r(R)$ of R . It is straightforward to verify that both $Soc_l(R)$ and $Soc_r(R)$ are two-sided ideals of R . In general the left socle of R need not equal its right socle (for instance, $Soc_l(R) \neq Soc_r(R)$ for R the ring of $n \times n$ upper triangular matrices over a field, $n \geq 2$). However, for rings R whose Jacobson radical is zero, it is known (see e.g. [18]) that $Soc_l(R) = Soc_r(R)$; in this situation, we shall denote the socle simply by $Soc(R)$.

The Jacobson radical $J(L_K(E))$ is shown to be $\{0\}$ in [10] and [11] for row-finite or countable graphs E . As we shall demonstrate in Proposition 1.8, in fact $J(L_K(E)) = \{0\}$ for any arbitrary graph E . In particular, we get $Soc_l(L_K(E)) = Soc_r(L_K(E))$. (We note that the equality $Soc_l(L_K(E)) = Soc_r(L_K(E))$ will also follow from the internal description of the socle given in Theorem 1.11.)

Definition 1.1. Let R be any (not necessarily unital) ring for which $J(R) = \{0\}$, and let $\tau = 2^{|R|}$. The *ascending Loewy socle series* of R is a well-ordered ascending chain of two sided ideals

$$0 = S_0 < S_1 < \cdots < S_\alpha < S_{\alpha+1} < \cdots \quad (\alpha < \tau)$$

where, for each $\alpha < \tau$,

$$S_{\alpha+1}/S_\alpha = Soc(R/S_\alpha) \text{ if } \gamma = \alpha + 1 \text{ is not a limit ordinal, and}$$

$$S_\gamma = \cup_{\alpha < \gamma} S_\alpha \text{ if } \gamma \text{ is a limit ordinal.}$$

For each $\alpha < \tau$, S_α is called the α -th *socle* of R . The least ordinal λ for which $S_\lambda = S_{\lambda+1}$ is called the *Loewy length* $\ell(R)$ of R . We call R a *Loewy ring* in case $R = S_\alpha$ for some α . We often refer to the ascending Loewy socle series of a ring simply as its *socle series*.

Using left (resp., right) socles one can define the notion of a left (resp., right) Loewy ring in the analogous manner.

For a unital ring R it is straightforward to show that R is a left Loewy ring if and only if every nonzero left R -module has a nonzero socle. Rings with this property have been studied under the name *left semiartinian rings* (see e.g. [14]).

We now review the appropriate graph- and algebra-theoretic terminology used in this article. For additional information about these ideas, see e.g. [1], [21], and [22].

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and maps $r, s : E^1 \rightarrow E^0$. (The sets E^0 and E^1 are allowed to be of arbitrary cardinality.) The elements of E^0 are called *vertices* and the elements of E^1 *edges*. A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source* of μ , $r(\mu) := r(e_n)$ is the *range* of μ , and n is the *length* of μ . We view the elements of E^0 as paths of length 0. If μ is a path in E , and if $v = s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$

for every $i \neq j$, then μ is called a *cycle based at v* . If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. We call a vertex v a *sink* if $s^{-1}(v)$ is empty; an *infinite emitter* if $s^{-1}(v)$ is an infinite set; and a *regular vertex* otherwise. A subset H of E^0 is said to be *hereditary* if whenever $u \in H$ and there is a path μ with $s(\mu) = u$ and $r(\mu) = w$ then $w \in H$. A subset H of E^0 is said to be *saturated* if, for any regular vertex $v \in E^0$, whenever $r(s^{-1}(v)) \subseteq H$ then $v \in H$. It is easy to see that the intersection of hereditary saturated subsets is again hereditary saturated. Given a hereditary subset H , the smallest hereditary saturated subset containing H is called the *saturated closure* of H , and is denoted by \overline{H} .

For any vertex $v \in E^0$, the *tree of v* is the set

$$T(v) = \{w \in E^0 : \text{there is a path } \mu \text{ with } s(\mu) = v \text{ and } r(\mu) = w\}.$$

(We note that $T(v)$ need not be a tree in the usual graph-theoretic sense, as $T(v)$ might contain cycles.) A vertex v is said to *have a bifurcation*, or *is a bifurcation vertex*, if $|s^{-1}(v)| > 1$. Following [10] and [11], we say a vertex v is a *line point* if no vertex in $T(v)$ is either a bifurcation vertex or is the base of a cycle. If v is a line point, then it is clear that every $w \in T(v)$ is also a line point; specifically, the set L of line points in a graph E is a hereditary subset. Observe that v is a line point exactly when the vertices in $T(v)$ form a (finite or infinite) line segment.

A ring R is said to be *von Neumann regular* if for each $x \in R$ there exists $y \in R$ for which $x = xyx$. A ring R is said to be \mathbb{Z} -graded in case there is a decomposition $R = \bigoplus_{n \in \mathbb{Z}} R_n$ as abelian groups having the property that $R_m \cdot R_n \subseteq R_{m+n}$ for all $m, n \in \mathbb{Z}$. For a \mathbb{Z} -graded ring R and $0 \neq x \in R$, write $x = \sum x_n$ with $x_n \in R_n$ for each n . The *degree* of x is the maximum n for which $x_n \neq 0$. An ideal I of a \mathbb{Z} -graded ring R is *graded* in case $I = \bigoplus_{n \in \mathbb{Z}} (I \cap R_n)$; equivalently, I is graded if whenever $x_n \in R_n$ for each n has $\sum x_n \in I$, then $x_n \in I$ for each n . In this case, observe that the ring R/I inherits a natural \mathbb{Z} -grading, and that the natural map $R \rightarrow R/I$ is a graded homomorphism. Moreover, for an ideal J of R having $J \supseteq I$, J is a graded ideal of R if and only if J/I is a graded ideal of R/I .

Definition 1.2. Let E be any directed graph, and K any field. The *Leavitt path K -algebra* $L_K(E)$ of E with coefficients in K is the K -algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* : e \in E^1\}$, which satisfy the following relations:

- (1) $s(e)e = er(e) = e$ for all $e \in E^1$.
- (2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
- (3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
- (4) $v = \sum_{\{e \in E^1 : s(e)=v\}} ee^*$ for every regular vertex $v \in E^0$.

When the role of K is not central to the discussion, we sometimes denote $L_K(E)$ simply by $L(E)$.

For any $e \in E^1$ we let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

Remark 1.3. In particular, by Property (3) we have that if w is any vertex in E , and μ is any path for which $r(\mu) = w$, then $\mu^*\mu = w$. On the other hand, if v is a vertex having only one edge e in $s^{-1}(v)$, then Property (4) yields $ee^* = v$. This observation and an easy

induction yields the following important property of the line points of E : if v is a line point, and μ is a path for which $s(\mu) = v$, then $\mu\mu^* = v$.

If E is a graph for which E^0 is finite then we have $\sum_{v \in E^0} v$ is the multiplicative identity in $L_K(E)$; otherwise, $L_K(E)$ is a ring with a set of local units consisting of sums of distinct vertices. In particular, if I is any ideal of $L_K(E)$, then $I = L_K(E)$ if and only if $I \cap E^0 = E^0$. Conversely, if $L_K(E)$ is unital, then E^0 is finite.

$L_K(E)$ is a \mathbb{Z} -graded K -algebra, spanned as a K -vector space by $\{pq^* : p, q \text{ are paths in } E\}$. (Recall that the elements of E^0 are viewed as paths of length 0, so that this set includes elements of the form v with $v \in E^0$.) In particular, for each $n \in \mathbb{Z}$, the degree n component $L_K(E)_n$ is spanned by elements of the form $\{pq^* : \text{length}(p) - \text{length}(q) = n\}$.

In the following result we present a useful description of the hereditary saturated closure of a hereditary subset of a graph.

Lemma 1.4. *Let H be a hereditary subset of vertices in a graph E . Then a vertex u belongs to the saturated closure \overline{H} of H if and only if there exists a positive integer n such that every path of length $\geq n$ in E that begins with u must end in a vertex of H .*

Proof. By definition, $\overline{H} = \bigcup_{n < \omega} H_n$ where H_n is defined inductively as follows. Let $H_0 = H$. If H_k has been defined for some $k \geq 0$, then set

$$H_{k+1} = H_k \cup \{v \in E^0 : v \text{ is a regular vertex, and } r(s^{-1}(v)) \subset H_k\}.$$

We show, by induction on k , that $v \in H_k$ if and only if every path of length $\geq k$ that begins with v must end in a vertex belonging to H . This clearly holds when $k = 0$, since $H_0 = H$ is hereditary. Assume we have proved the result when $k = m \geq 0$. Let $v \in E^0$ be a regular vertex, say $s^{-1}(v) = \{e_1, \dots, e_t\}$ and $r(e_i) = v_i$, for $i = 1, \dots, t$. Now $v \in H_{m+1}$ precisely when $v \in H_m$ or when $r(s^{-1}(v)) = \{v_1, \dots, v_t\} \subset H_m$. By induction, this is equivalent to saying that every path of length at least m that begins with any vertex v_i ends in H . This is clearly equivalent to requiring that every path of length at least $m+1$ that begins with v must end in a vertex belonging to H . Hence the result follows. \square

We continue by describing the following concepts which were initially introduced in papers dealing with C^* -algebras (see e.g. [21]). These ideas were also considered for row-finite graphs in [12], and for not-necessarily row-finite graphs in [22].

Definition 1.5. Let E be an arbitrary graph and let H be a hereditary saturated subset of vertices in E .

(i) The set of *breaking vertices* of H , denoted B_H , is the set

$$B_H = \{v \in E^0 \setminus H : v \text{ is an infinite emitter, and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

(ii) The *quotient graph* $E|H$ is defined as follows. Let B'_H be a set which is in one-to-one correspondence with B_H , and write $B'_H = \{v' : v \in B_H\}$. Define

$$(E|H)^0 = (E^0 \setminus H) \cup B'_H \quad \text{and} \quad (E|H)^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1 \text{ with } r(e) \in B_H\}.$$

The source and range functions $s_{E|H}$ and $r_{E|H}$ coincide with the functions s_E and r_E when applicable, while we define $s_{E|H}(e') = s_E(e)$ and $r_{E|H}(e') = (r_E(e))'$.

We note that each v' is a sink in the graph $E|H$, and so is a line point in $E|H$.

A useful tool in our construction is the following important theorem of Tomforde ([22, Theorem 5.7]). This theorem has been established under the hypothesis that E is a graph with at most countably many vertices and edges; however, an examination of the proof reveals that the countability condition on E is not utilized. Hence we give a reformulation, tailored to our needs, of various parts of [22, Lemma 5.6 and Theorem 5.7] for arbitrary graphs E .

Definition 1.6. Let E be a graph, and S any subset of E^0 . We denote by $I(S)$ the two-sided ideal of $L(E)$ generated by S .

Theorem 1.7. *Let E be an arbitrary graph, and let H be a hereditary saturated subset of E . Then*

- (i) $I(H)$ is a graded ideal of $L_K(E)$.
- (ii) There is an algebra epimorphism

$$\phi : L_K(E) \rightarrow L_K(E|H)$$

for which $\ker \phi = I(H)$. In particular,

$$L_K(E)/I(H) \cong L_K(E|H).$$

Moreover, for each $v \in B_H$ we have $\phi(v - \sum_{s(e)=v, r(e) \notin H} ee^*) = v'$ (where v' is described in Definition 1.5).

- (iii) Let I be a graded ideal of $L_K(E)$. If $I \cap E^0 = H$, and we define

$$S = \{v \in B_H : v - \sum_{s(e)=v, r(e) \notin H} ee^* \in I\},$$

then $I = I(H \cup S)$.

As an application of Theorem 1.7, we get as promised the following extension to arbitrary graphs of a result proved in [3] and [11] for row-finite or countable graphs.

Proposition 1.8. *Let E be an arbitrary graph. Then the Jacobson radical $J(L(E)) = \{0\}$. In particular, $L(E)$ contains no nonzero nilpotent left or right ideals.*

Proof. Since $L(E)$ is a ring with local units, $J = J(L(E))$ is a graded ideal by [3, Lemma 6.2]. From Theorem 1.7(iii), J is the ideal of $L(E)$ generated by $H \cup S$, where $H = E^0 \cap J$ and $S = \{v \in B_H : v - \sum_{s(e)=v, r(e) \notin H} ee^* \in J\}$. Since J contains no nonzero idempotents, H must be the empty set. From the definition of B_H we then conclude that B_H , and hence S , must be the empty set as well. This implies that $J = \{0\}$. \square

When $R = L_K(E)$ and E is row-finite or is countably infinite, then $\text{Soc}(R)$ has been described in [10] and [11]. Specifically, $\text{Soc}(R)$ is the ideal of $L_K(E)$ generated by the line points of E . Our goal for the remainder of this section is to extend this result to the case where E is an arbitrary graph. Along the way, we provide a simpler proof of the fact that the line points are precisely those vertices which generate a simple left (equivalently, right) ideal of $L_K(E)$.

Proposition 1.9. ([10], [11]) *Let E be an arbitrary graph and $v \in E^0$. Then v is a line point exactly when the left ideal $L_K(E)v$ of $L_K(E)$ is simple.*

Proof. Suppose v is a line point. We claim that every nonzero $L_K(E)$ -endomorphism of $L_K(E)v$ is an automorphism. Since v is an idempotent, it is well-known (and can also be easily seen) that $\text{End}(L_K(E)v) \cong vL_K(E)v$. An arbitrary element $a \in vL_K(E)v$ will be of the form $a = v(\sum_{i=1}^n k_i p_i q_i^*)v = \sum_{i=1}^n k_i (vp_i q_i^* v)$. Observe that if $vp_i q_i^* v \neq 0$ for some $1 \leq i \leq n$, then $s(p_i) = s(q_i) = v$, and $r(p_i) = r(q_i)$. Since v is a line point, this yields that $p_i = q_i$. But by Remark 1.3 we then get $vp_i q_i^* v = vp_i p_i^* v = v$. Hence $a = (\sum_{i=1}^n k_i)v$ and we conclude that $vL_K(E)v = Kv$ is a field with identity element v , which establishes the claim.

Since $L_K(E)$ has local units, we have $L_K(E)a \neq \{0\}$. By Proposition 1.8, $L_K(E)$ has no nonzero nilpotent one-sided ideals, so that $(L_K(E)a)^2 \neq \{0\}$. In particular there is an element $b \in L_K(E)$ such that $aba \neq 0$. Thus right multiplication by ba is a nonzero endomorphism of $L_K(E)v$ (as $aba \neq 0$), and hence by the previous paragraph is an automorphism. In particular $v = cba$ for some $c \in L_K(E)$. This means that $v \in L_K(E)a$, showing that $L_K(E)v$ is simple.

Conversely, suppose $L_K(E)v$ is simple. Suppose by way of contradiction that $T(v)$ has vertices with bifurcations, and choose a bifurcation vertex $u \in T(v)$ so that there is a path μ of shortest length connecting v to u . Then $L_K(E)u$ is simple, since using Remark 1.3 it is easy to show that right multiplication by μ induces an isomorphism from $L_K(E)v$ to $L_K(E)u$ (with inverse map right multiplication by μ^*). As shown in [10, Proposition 2.5], the existence of a cycle based at a vertex in $T(v)$ would contradict the simplicity of $L_K(E)v$, so we may assume that $T(v)$ is acyclic. Now let e be an edge with $s(e) = u$. Then $L_K(E)u = L_K(E)ee^* \oplus C$, where $C = \{x - xee^* : x \in L_K(E)\}$. Now $C \neq \{0\}$, since there is another edge $f \neq e$ with $s(f) = u$ (and $fe = 0$ since f is not a loop by the previous observation), so that $f = f - fee^* \in C$. This contradicts the simplicity of $L_K(E)u$. Hence $T(v)$ contains no vertices with bifurcations. Hence v must be a line point. \square

Remark 1.10. Although it would also follow from more general (and much deeper) structural results about semiprime rings, in fact a direct “left to right” modification of the proof given above easily yields that $v \in E^0$ is a line point if and only if the right ideal $vL_K(E)$ of $L_K(E)$ is simple.

The following result is established in [10, Theorem 4.2] for row-finite graphs, and in [11, Theorem 5.2] for countable graphs; effectively the proof is the same in both cases. Using Proposition 1.9, indeed this same proof can be used to establish the result for arbitrary graphs E . That is, we have

Theorem 1.11. *For an arbitrary graph E and field K , $\text{Soc}(L_K(E))$ is the two sided ideal generated by the set of line points in E .*

Theorems 1.7 and 1.11 provide us with the two fundamental tools we will use to establish our main results. We seek to understand the behavior of the socle series of $L(E)$; this requires us to understand the behavior of the socle of each of the quotients $L(E)/S_\alpha$. By Theorem 1.7 we will be able to realize $L(E)/S_\alpha$ as $L(F)$ for some graph F . In turn, by Theorem 1.11, we will be able to identify the socle of this quotient in terms of the line points of F .

2. EXAMPLES

In this section we analyze the Loewy lengths of Leavitt path algebras $L_K(E)$ of various graphs E over any field K . We begin with the most basic type of Leavitt path algebras.

Example 2.1. Let E be a finite acyclic graph. Then $L_K(E)$ is semisimple artinian, as $L_K(E)$ is a direct sum of complete matrix rings over K (see e.g. [5, Proposition 3.5]). Thus we have in this case that $L_K(E)$ is a Loewy ring, and $\ell(L_K(E)) = 1$. In particular, the trivial graph

$$P_0 : \quad \bullet^v$$

consisting of one vertex and no edges is a Loewy ring of length 1.

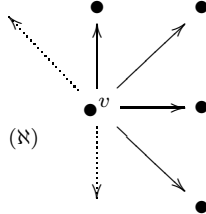
Example 2.2. In this second example we present Leavitt path algebras which are not Loewy rings. First, if E is the graph with a single vertex v and a single loop x at v , then $R = L_K(E) \cong K[x, x^{-1}]$, the ring of Laurent polynomials over K . In this case $\ell(R) = 0$ since $\text{Soc}(R) = \{0\}$. Seen another way, since E has no line points, $\text{Soc}(R) = \{0\}$ by Theorem 1.11. In particular, R is not a Loewy ring.

Now let T be the graph

$$\begin{array}{c} \textcirclearrowleft \\ \bullet^v \longrightarrow \bullet^u \end{array}$$

Since u is the unique line point of T , we get that $R = L_K(T)$ satisfies $S_1 = \text{Soc}(R) = I(u)$, the ideal generated by u . It is easy to show that $\text{Soc}(R) \neq R$, since in particular $v \notin \text{Soc}(R)$. Also, by Theorem 1.7(ii), $R/S_1 \cong L_K(T \setminus \{u\})$. Since $T \setminus \{u\}$ is the graph with a single vertex and single edge, we obtain from the preceding paragraph that $R/S_1 \cong K[x, x^{-1}]$ and that $\text{Soc}(R/S_1) = \{0\}$. Hence $S_1 = S_2 = \dots$. Specifically, we have $\ell(R) = 1$, but R is not a Loewy ring as $R \neq S_i$ for any i .

Example 2.3. Let \aleph be any infinite cardinal, and let C_\aleph be the “infinite clock” graph with \aleph edges



The set $H = \{r(e) : e \in (C_\aleph)^1\}$ is a hereditary saturated subset of $(C_\aleph)^0$ (recall that by definition the saturated condition only applies at regular vertices), and indeed is precisely the set of line points in C_\aleph . By Theorem 1.11, $\text{Soc}(L_K(C_\aleph)) = I(H)$. Then, by Theorem 1.7(ii), $L_K(C_\aleph)/I(H) \cong L_K(C_\aleph|H) = L_K(\{v\}) \cong K$. Since $\text{Soc}(K) = K$, we get that $L_K(C_\aleph)$ is thus a Loewy ring with Loewy length 2.

Example 2.4. Let P_0^ω denote the graph consisting of countably many vertices and no edges. Then $L_K(P_0^\omega) \cong \bigoplus_{n \in \mathbb{N}} K$ as (nonunital) rings, so that we immediately conclude that $L_K(P_0^\omega) = \text{Soc}(L_K(P_0^\omega))$. In particular, $L_K(P_0^\omega)$ is a Loewy ring of Loewy length 1. (Note that every vertex in P_0^ω is vacuously a line point, so that this result also follows immediately from Theorem 1.11.)

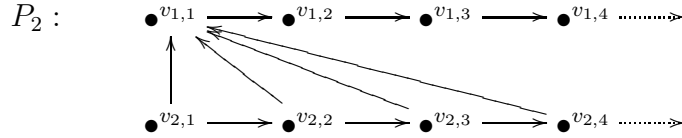
Example 2.5. Let P_1 denote the “infinite line” graph

$$P_1 : \quad \bullet^{v_{1,1}} \longrightarrow \bullet^{v_{1,2}} \longrightarrow \bullet^{v_{1,3}} \longrightarrow \bullet^{v_{1,4}} \cdots \longrightarrow$$

Then clearly every vertex in P_1 is a line point, so by Theorem 1.11 we conclude that $L_K(P_1) = \text{Soc}(L_K(P_1))$. In particular, $L_K(P_1)$ is a Loewy ring of Loewy length 1.

The graphs P_0 and P_1 of the previous examples will be used as the foundation for the construction presented in the proof of Theorem 4.1. Intuitively, P_0 and P_1 provide the two canonical examples of graphs for which the corresponding Leavitt path algebras are Loewy rings of Loewy length 1.

Example 2.6. Let P_2 denote the graph

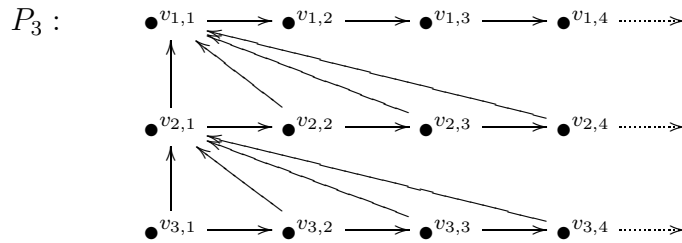


The set $H = \{v_{1,j} : j \in \mathbb{N}\}$ (i.e., the “top row” of P_2) is the set of line points in the graph P_2 ; furthermore, an easy observation yields that $H = \overline{H}$. Note that the quotient graph $P_2|H$ consists of the vertices and edges in the “bottom row” of P_2 , which is clearly isomorphic as a graph to P_1 . Now by Theorem 1.11, $\text{Soc}(L_K(P_2)) = I(H)$, the ideal generated by H . Furthermore, by Theorem 1.7(ii),

$$L_K(P_2)/\text{Soc}(L_K(P_2)) = L_K(P_2)/I(H) \cong L_K(P_2|H) \cong L_K(P_1).$$

We conclude that $L_K(P_2)$ is a Loewy ring having $\ell(L_K(P_2)) = 2$.

Example 2.7. For each integer $i \geq 2$ we construct inductively the “pyramid” graph P_i as follows. The graphs P_1 and P_2 are presented in the preceding examples. For each $i \geq 2$ we construct the graph P_{i+1} from the graph P_i by adding vertices $v_{i+1,1}, v_{i+1,2}, v_{i+1,3}, \dots$ and two sets of edges: for each $j \in \mathbb{N}$, an edge from $v_{i+1,j}$ to $v_{i+1,j+1}$; and, for each $j \geq 1$, a single edge from $v_{i+1,j}$ to $v_{i,1}$. So, for example, we have



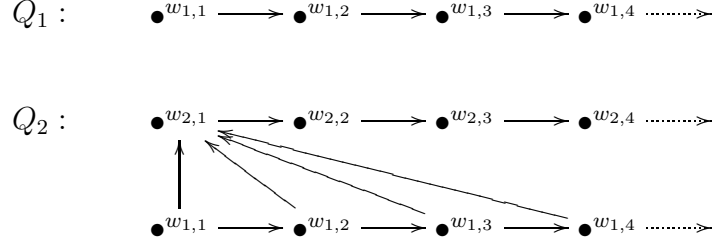
By induction, using the argument of Example 2.6, it is straightforward to show, for each $i \in \mathbb{N}$, that $L_K(P_i)$ is a Loewy ring for which $\ell(L_K(P_i)) = i$.

We now view $P_i \subseteq P_{i+1}$ for each $i \in \mathbb{N}$, and make two observations. First, it is clear that P_i^0 (the set of vertices of P_i) is a hereditary saturated subset of P_{i+1} . (Notice that although each vertex $v_{i+1,n}$ emits an edge into P_i^0 , each such vertex as well emits an edge whose range vertex is *not* in P_i^0 , whereby the saturated property of P_i^0 follows.) In particular we can form the quotient graph $P_{i+1}|P_i^0$; it is immediate that $P_{i+1}|P_i^0 \cong P_1$ as graphs. Second, with this inclusion of graphs we can then form the graph

$$P_\omega = \cup_{i < \omega} P_i.$$

Again invoking the argument of Example 2.6, one can similarly show that $L_K(P_\omega)$ is a Loewy ring, and $\ell(L_K(P_\omega)) = \omega$.

Example 2.8. We consider here a construction which looks quite similar to that achieved in Example 2.7, but which features an interesting twist. For each $i \in \mathbb{N}$ we define the graph Q_i as pictured here.



For each $i \geq 2$ we construct the graph Q_{i+1} from the graph Q_i by adding vertices $w_{i+1,1}, w_{i+1,2}, w_{i+1,3}, \dots$ and two sets of edges, as follows. For each $j \in \mathbb{N}$, we add an edge from $w_{i+1,j}$ to $w_{i+1,j+1}$; and, for each $j \geq 1$, we add a single edge from $w_{i,j}$ to $w_{i+1,1}$. Clearly for each $i \in \mathbb{N}$ the graph Q_i is isomorphic to the graph P_i of Example 2.7. In particular, for each $i \in \mathbb{N}$ we have that $L_K(Q_i)$ is a Loewy ring, and $\ell(L_K(Q_i)) = i$.

Here is where the two examples diverge. If we view $Q_i \subseteq Q_{i+1}$ for each $i \in \mathbb{N}$, then we can form the graph $Q_\omega = \cup_{i < \omega} Q_i$. Unlike its counterpart P_ω , the graph Q_ω contains no line points. In particular, $\text{Soc}(L_K(Q_\omega)) = \{0\}$, so that $S_\alpha = \{0\}$ for all α . Thus, unlike its counterpart $L_K(P_\omega)$, the Leavitt path algebra $L_K(Q_\omega)$ is not a Loewy ring.

Remark 2.9. For an arbitrary ring R , dual to the left (resp., right) ascending Loewy socle series is the corresponding left (resp., right) *descending Loewy radical series*. Briefly, this is defined by setting R_0 to be R , setting $R_{\alpha+1}$ to be the intersection of all maximal left (resp., right) submodules of R_α , and, for any limit ordinal γ , setting R_γ to be $\cap_{\alpha < \gamma} R_\alpha$. (If R_α has no maximal submodules, then set $R_\alpha = R_{\alpha+1}$.) Specifically, $R_1 = J(R)$. But $J(L_K(E)) = \{0\}$ for any graph E by Proposition 1.8, so that $R_1 = \{0\}$. Thus the descending Loewy radical series is of little interest in the context of Leavitt path algebras.

3. LEAVITT PATH ALGEBRAS OF ARBITRARY LOEWY LENGTH

The goal of this section is to describe in graph-theoretic terms a necessary and sufficient condition on an arbitrary graph E in order that the corresponding Leavitt path algebra $L_K(E)$ is a Loewy ring; we achieve this characterization in Theorem 3.2. As a result of this description, we are able to construct, for each ordinal λ , a graph E with the property that $L_K(E)$ is a Loewy ring of length λ (Theorem 4.1).

Recall that for a hereditary subset S of vertices of a graph E , the set \overline{S} denotes the saturated closure of S in E^0 .

Definition 3.1. Let E be an arbitrary graph and consider the Leavitt path algebra $L_K(E)$. For each ordinal γ we define transfinitely a hereditary saturated subset V_γ of E^0 as follows.

- (1) V_1 is the saturated closure of the set of line points of E .

Suppose $\gamma > 1$ is any ordinal and that the sets V_α have been defined for all $\alpha < \gamma$.

- (2) If γ is a limit ordinal, then $V_\gamma = \cup_{\alpha < \gamma} V_\alpha$.

(3) If $\gamma = \alpha + 1$ is a non-limit ordinal, then $V_\gamma = E^0 \cap I$, where I is the ideal of $L_K(E)$ generated by the set

$$\begin{aligned} V_\alpha \cup \{ & w \in E^0 \setminus V_\alpha : \text{every bifurcation vertex } u \in T_E(w) \setminus V_\alpha \text{ has at most one edge } e \\ & \text{with } s(e) = u \text{ and } r(e) \notin V_\alpha \} \\ \cup \{ & v - \sum_{\substack{s(e)=v \\ r(e) \notin V_\alpha}} ee^* : v \in B_{V_\alpha} \}. \end{aligned}$$

Theorem 3.2. *Let E be an arbitrary graph and K any field. For each ordinal α let S_α denote the α -th socle of $L_K(E)$, and let V_α denote the subset of E^0 given in Definition 3.1. Then*

- (1) S_α is a graded ideal of $L_K(E)$ for each α .
- (2) $V_\alpha = E^0 \cap S_\alpha$ for each α .
- (3) $L_K(E)/S_\alpha \cong L(E|V_\alpha)$ as graded K -algebras for each α .
- (4) $L_K(E)$ is a Loewy ring of length λ if and only if λ is the smallest ordinal such that $E^0 = V_\lambda$.

Proof. We establish statements (1) and (2) simultaneously by transfinite induction.

When $\gamma = 1$, V_1 has been defined to be the saturated closure of the set of all line points in E ; so V_1 is a hereditary saturated subset of E^0 . By Theorem 1.11, $S_1 = \text{Soc}(L_K(E))$ is the two-sided ideal of $L_K(E)$ generated by $V_1 = E^0 \cap S_1$, and hence is a graded ideal of $L_K(E)$ by Theorem 1.7(i).

Now suppose $\gamma > 1$ and, for all $\alpha < \gamma$, we have that $V_\alpha = E^0 \cap S_\alpha$ and that S_α has been shown to be a graded ideal. Recall (see Definition 1.5) that $E|V_\alpha$ is the graph with

$$(E|V_\alpha)^0 = (E^0 \setminus V_\alpha) \cup B'_{V_\alpha} \quad \text{and} \quad (E|V_\alpha)^1 = \{e \in E^1 : r(e) \notin V_\alpha\} \cup \{e' : e \in E^1 : r(e) \in B_{V_\alpha}\}.$$

Also, $s(e), r(e)$ are defined as in E if $e \in E^1$, while $s(e') = s(e)$ and $r(e') = (r(e))'$. Then by Theorem 1.7(ii), there is an epimorphism $\phi : L_K(E) \rightarrow L_K(E|V_\alpha)$ having $\ker(\phi) = S_\alpha$, for which $L_K(E)/S_\alpha \cong L_K(E|V_\alpha)$.

Suppose γ is not a limit ordinal (so that $\gamma = \alpha + 1$ for some α), and suppose $V_\alpha \neq E^0$ (and thus $S_\alpha \neq L_K(E)$). Then define

$$\begin{aligned} V'_{\alpha+1} = \{ & w \in E^0 \setminus V_\alpha : \text{every bifurcation vertex } u \in T_E(w) \setminus V_\alpha \text{ has at most one edge } e \\ & \text{with } s(e) = u \text{ and } r(e) \notin V_\alpha \} \\ \cup \{ & v - \sum_{\substack{s(e)=v \\ r(e) \notin V_\alpha}} ee^* : v \in B_{V_\alpha} \}. \end{aligned}$$

From Theorem 1.7(ii) we have that

$$\phi(v - \sum_{s(e)=v, r(e) \notin V_\alpha} ee^*) = v'.$$

As noted previously, v' is a sink and hence a line point in $E|V_\alpha$. It is then easy to check that $\phi(V'_{\alpha+1})$ is exactly the set of line points of $E|V_\alpha$. So if I is the ideal of $L_K(E)$ generated by $S_\alpha \cup V'_{\alpha+1}$, then we get that $\text{Soc}(L_K(E|V_\alpha)) \cong I/S_\alpha = \text{Soc}(L_K(E)/S_\alpha)$. Thus $I = S_{\alpha+1}$, the $\alpha + 1$ -st socle of $L_K(E)$. Since S_α and $S_{\alpha+1}/S_\alpha = \text{Soc}(L_K(E)/S_\alpha)$ are each graded ideals, so is $S_{\alpha+1}$ by a previous observation. But then $V_{\alpha+1} = E^0 \cap I$ by construction.

Thus we have verified (1) and (2) in the induction process whenever γ is not limit ordinal.

On the other hand, suppose γ is a limit ordinal. Then by definition $V_\gamma = \cup_{\alpha < \gamma} V_\alpha$, and $S_\gamma = \cup_{\alpha < \gamma} S_\alpha$. Since each S_α is a graded ideal, so is S_γ . It is then immediate that $V_\gamma = E^0 \cap S_\gamma$.

Thus we have established (1) and (2) by transfinite induction. But then (3) follows from (1) and (2) together with Theorem 1.7(ii). Finally, (4) follows immediately from (2), since for any ideal I of $L_K(E)$, $I = L_K(E)$ if and only if $I \cap E^0 = E^0$. \square

We can in fact glean from the proof of Theorem 3.2 some additional information about the individual members of the ascending Loewy socle series of a Leavitt path algebra.

Proposition 3.3. *Each S_α is a von Neumann regular ring.*

Proof. It is known (see e.g. [18, pages 65, 90]) that if R is a semiprime ring (i.e., R has no nonzero nilpotent ideals), then the socle $Soc(R)$ is a direct sum of simple rings T_i , each of which is a direct sum of isomorphic simple left ideals and, moreover, each T_i is a directed union of full matrix rings over division rings. In particular each T_i , and hence $Soc(R)$, is von Neumann regular. Thus, by Proposition 1.8, for the Leavitt path algebra $L_K(E)$, its socle S_1 is always von Neumann regular. From the proof of Theorem 3.2, we notice the α -th socle S_α is generated by the hereditary saturated set V_α and that, by Theorem 1.7(ii), $L_K(E)/S_\alpha$ is again a Leavitt path algebra (as it is isomorphic to $L_K(E|V_\alpha)$). This implies that $L_K(E)/S_\alpha$ is semiprime since, by Proposition 1.8, $J(L(E|V_\alpha)) = 0$. Consequently, $S_{\alpha+1}/S_\alpha = Soc(L_K(E)/S_\alpha)$ is von Neumann regular for all $\alpha \geq 1$. Since an extension of a von Neumann regular ring by another von Neumann regular ring is again von Neumann regular, and since von Neumann regularity survives under ascending unions, we conclude, by transfinite induction, that each S_α is von Neumann regular. \square

Recall that a *K-matrical algebra* is a finite direct product of full matrix algebras over K , while a *locally K-matrical algebra* is a direct limit of *K-matrical algebras*. As a consequence of Proposition 3.3, [6, Theorem 1] then gives

Corollary 3.4. *Let E be an arbitrary graph. If $L_K(E)$ is a Loewy ring, then necessarily E must be acyclic and $L_K(E)$ must be locally *K-matrical* (and, in particular, von Neumann regular).*

We note that an arbitrary Loewy ring R need not be von Neumann regular even if it is semiprime, as shown for instance in [14, Example 2.2].

As Example 2.8 yields the acyclic graph Q_ω for which $\ell(L_K(Q_\omega)) = 0$, we see that the converse to Corollary 3.4 does not hold in general. However,

Corollary 3.5. *Let E be a graph for which E^0 is finite, and K any field. The following are equivalent.*

- (1) $L_K(E)$ is a Loewy ring.
- (2) E is acyclic.
- (3) $L_K(E)$ is von Neumann regular.

If in addition E^1 is also finite, then the previous conditions are equivalent to

- (4) $L_K(E)$ is semisimple artinian. (In particular, in this case we have $\ell(L_K(E)) = 1$.)

Proof. (1) implies (2) follows from Corollary 3.4. For (2) implies (1), since E^0 is finite and E is acyclic then necessarily E contains sinks. But any sink is necessarily a line point in E . So $S_1 = \text{Soc}(L(E)) \neq \{0\}$. If $S_1 \neq L(E)$ then by Theorem 1.7 the quotient $L(E)/S_1 \cong L(F)$ for some (necessarily acyclic) quotient graph F of E , for which $|F^0| < |E^0|$. An induction argument now gives the result. The equivalence of (2) and (3) was established in [6, Theorem 1].

In case E^1 is also finite, the equivalence of conditions (2), (3), and (4) follows from the fact (see e.g. [5, Proposition 3.5]) that $L_K(E)$ in this situation is isomorphic to a finite direct sum of full matrix rings over K . That $\ell(L_K(E)) = 1$ in this case then follows directly from (4). \square

4. LEAVITT PATH ALGEBRAS OF PRESCRIBED LOEWY LENGTH

In this final section we demonstrate, as a key consequence of Theorem 3.2, how to construct graphs E for which the corresponding Leavitt path algebras $L_K(E)$ are Loewy rings of arbitrarily prescribed Loewy length.

The idea of the following construction is to build graphs for which the subsets V_λ are well understood. Intuitively, we do this by ensuring that for each α , the quotient $L_K(E)/S_\alpha$ is isomorphic to the Leavitt path algebra of either the graph P_0 or the graph P_1 . As noted previously, these two graphs are the prototypical graphs whose Leavitt path algebras are Loewy rings of length 1. (In fact these two graphs are related to each other: clearly the single vertex of P_0 is a sink in the usual sense, while the graph P_1 may naturally be viewed as an *infinite sink*. See e.g. [4, Definition 1.7].)

Theorem 4.1. *For every ordinal λ and any field K , there is an acyclic graph P_λ for which $L_K(P_\lambda)$ is a Loewy ring of length λ .*

Proof. For $\lambda = 1$, choose $E = P_1$, the “infinite line” graph of Example 2.5. Observe that P_1 is acyclic, and has countably many vertices, each of which is a line point. Thus, in the notation of Theorem 3.2, $V_1 = P_1^0 = E^0$.

We now utilize an approach similar to the one used in Example 2.7 to construct transfinitely the graphs $P_{\alpha+1}$ from P_α for various ordinals α . This construction agrees with the construction of the graphs P_n of Example 2.7 for n finite.

Suppose $\gamma \geq 2$ is any ordinal and assume that the graphs P_α have already been defined for all $\alpha < \gamma$ in such a way that:

- (1) P_α is acyclic,
- (2) P_α is a subgraph of $P_{\alpha+1}$ for all $\alpha + 1 < \gamma$, and
- (3) whenever α is not a limit ordinal, $P_{\alpha-1}^0$ (i.e., the vertices of $P_{\alpha-1}$) is a hereditary saturated subset of P_α , and the quotient graph $P_\alpha|P_{\alpha-1}^0$ is isomorphic as a graph to either the graph P_0 or the graph P_1 given in Examples 2.1 and 2.5.

There are three possibilities for γ .

First, if γ is a limit ordinal, then define

$$P_\gamma = \cup_{\alpha < \gamma} P_\alpha.$$

Second, suppose $\gamma = \alpha + 1$, where α is a limit ordinal. By definition, P_α is the union of P_β , $\beta < \alpha$. We define the graph $P_\gamma = P_{\alpha+1}$ by introducing new symbols $v_{\alpha+1,1}$ and $\{e_\beta^{\alpha+1} : \beta < \alpha\}$, and by setting:

$$P_\gamma^0 = P_\alpha^0 \cup \{v_{\alpha+1,1}\}, \text{ and } P_\gamma^1 = P_\alpha^1 \cup \{e_\beta^{\alpha+1} : \beta < \alpha\}$$

and by defining $s(e_\beta^{\alpha+1}) = v_{\alpha+1,1}$ and $r(e_\beta^{\alpha+1}) = v_{\beta,1}$.

Pictorially,

$$P_\gamma = P_{\alpha+1} = P_\alpha \cup \begin{array}{c} \uparrow \\ \{e_\beta^{\alpha+1}\} \\ \uparrow \\ \bullet v_{\alpha+1,1} \end{array}$$

where the double arrow indicates multiple edges indexed by $\beta < \alpha$, with $r(e_\beta^{\alpha+1}) = v_{\beta,1}$ for each β .

For the third possibility, suppose $\gamma = \alpha + 1$, where α is not a limit ordinal. We define $P_\gamma = P_{\alpha+1}$ by introducing new symbols $\{v_{\alpha+1,n} : n \in \mathbb{N}\}$, $\{e_{\alpha,n} : n \in \mathbb{N}\}$, and $\{f_{\alpha+1,n} : n \in \mathbb{N}\}$, and by setting:

$$P_\gamma^0 = P_\alpha^0 \cup \{v_{\alpha+1,n} : n \in \mathbb{N}\}, \text{ and } P_\gamma^1 = P_\alpha^1 \cup \{e_{\alpha,n} : n \in \mathbb{N}\} \cup \{f_{\alpha+1,n} : n \in \mathbb{N}\},$$

where $s(e_{\alpha,n}) = v_{\alpha+1,n}$, $r(e_{\alpha,n}) = v_{\alpha,1}$, $s(f_{\alpha+1,n}) = v_{\alpha+1,n}$, and $r(f_{\alpha+1,n}) = v_{\alpha+1,n+1}$ for all $n \in \mathbb{N}$.

Pictorially,

$$P_\gamma = P_{\alpha+1} = P_\alpha \cup \begin{array}{c} \begin{array}{c} \uparrow \\ e_{\alpha,1} \\ \bullet v_{\alpha+1,1} \end{array} \quad \begin{array}{c} \nearrow e_{\alpha,2} \\ \bullet v_{\alpha+1,1} \end{array} \quad \begin{array}{c} \nearrow e_{\alpha,3} \\ \bullet v_{\alpha+1,1} \end{array} \quad \begin{array}{c} \nearrow \dots \\ \bullet v_{\alpha+1,1} \end{array} \\ \bullet v_{\alpha+1,1} \xrightarrow{f_{\alpha+1,1}} \bullet v_{\alpha+1,2} \xrightarrow{f_{\alpha+1,2}} \bullet v_{\alpha+1,3} \xrightarrow{f_{\alpha+1,3}} \bullet \dots \end{array}$$

with $r(e_{\alpha,n}) = v_{\alpha,1}$ for all $n \in \mathbb{N}$.

By transfinite induction, the graphs P_γ are now defined for every ordinal γ . It is clear from our construction that each P_γ is an acyclic graph. Moreover, it is also clear from the construction that each P_γ satisfies the indicated conditions (2) and (3) above. (Again, recall that the saturated condition applies only for regular vertices; note that in the case where α is a limit ordinal, the vertex $\{v_{\alpha+1,1}\}$ is an infinite emitter, and thus not a regular vertex. Note also that in the graphs P_α the “breaking vertex” sets B_H are empty for all germane subsets H .)

Thus for each λ the graph P_λ has the property that $V_\lambda = P_\lambda^0$, and so P_λ is a graph of the desired type by Theorem 3.2(4). □

Recall that in Section 2 we constructed, for each $n \leq \omega$, a row-finite graph P_n for which $L(P_n)$ is a Loewy ring of length n . We finish this article by showing that, in the row-finite case, length ω is the maximum possible.

Theorem 4.2. *If E is a row-finite graph, then $L_K(E)$ must have Loewy length $\leq \omega$.*

Proof. Suppose, by way of contradiction, that $L_K(E)$ has length $> \omega$. Let S_ω be the ω -th socle of $L_K(E)$. By Theorem 1.7(iii), S_ω is the ideal of $L_K(E)$ generated by $V_\omega = \cup_{n < \omega} V_n$. For the same reason, $S_{\omega+1}$ is the ideal of $L_K(E)$ generated by $V_{\omega+1} = V_\omega \cup Z$, where, in general, Z is the union of two subsets of vertices (recall Definition 3.1). However, since E is row-finite, the second of these two subsets is empty, so that here we have

$$Z = \{v \in E^0 \setminus V_\omega : \text{every bifurcation vertex } u \in T_E(v) \setminus V_\omega \text{ has at most one edge } e \in E^1 \\ \text{with } s(e) = u \text{ and } r(e) \notin V_\omega\}.$$

Let $v \in Z$ and let $\{u_i : u_i \in T_E(v) \setminus V_\omega, i \in X\}$ be the set of all bifurcation vertices in $T_E(v) \setminus V_\omega$. For a given $i \in X$, let $s^{-1}(u_i) = \{e_{i_1}, \dots, e_{i_{k_i}}\}$, and let

$$J_i = \{e_{i_j} \in s^{-1}(u_i) : r(e_{i_j}) \in V_\omega\}.$$

Note that, by the conditions on $v \in Z$, $|J_i| = k_{i-1}$ or k_i . Since V_ω is the union of the ascending chain $V_1 \subset V_2 \subset \dots$, there is a positive integer m such that $r(J_i) \subseteq V_m$. Thus u_i is a line point in $E|V_m$. But by Theorem 3.2(3) we have $L_K(E)/S_m \cong L_K(E|V_m)$, and so u_i maps into the socle of $L_K(E)/S_m$. This means that

$$u_i \in S_{m+1} \cap E^0 = V_{m+1} \subset V_\omega.$$

This contradicts the fact that $u_i \in T_E(v) \setminus V_\omega$. Hence $L_K(E)$ must have length $\leq \omega$. \square

REFERENCES

- [1] G. ABRAMS AND G. ARANDA PINO, The Leavitt path algebra of a graph, *J. Algebra* **293**(2) (2005), 319–334.
- [2] G. ABRAMS AND G. ARANDA PINO, Purely infinite simple Leavitt path algebras, *J. Pure Appl. Algebra* **207**(3) (2006), 553–563.
- [3] G. ABRAMS AND G. ARANDA PINO, The Leavitt path algebras of arbitrary graphs, *Houston J. Math* **34**(2) (2008), 423–442.
- [4] G. ABRAMS, G. ARANDA PINO, F. PERERA, M. SILES MOLINA, Chain conditions for Leavitt path algebras, *Forum Math.*, to appear.
- [5] G. ABRAMS, G. ARANDA PINO, AND M. SILES MOLINA, Finite-dimensional Leavitt path algebras, *J. Pure App. Alg.* **209**(3) (2007), 753–762.
- [6] G. ABRAMS AND K.M. RANGASWAMY, Regularity conditions for arbitrary Leavitt path algebras, *Algebr. Represent. Theory*, to appear.
- [7] F.W. ANDERSON AND K.FULLER, Rings and categories of modules, Graduate Texts in Math., vol. 13, Springer - Verlag, New York - Berlin (1973).
- [8] P. ARA AND E. PARDO, Stable rank of Leavitt path algebras, *Proc. A.M.S.* **136** (2008), 2375–2386.
- [9] P. ARA, M.A. MORENO, AND E. PARDO, Nonstable K -theory for graph algebras, *Algebr. Represent. Theory* **10** (2007), 157–178.
- [10] G. ARANDA PINO, D. MARTIN BARQUERO, C. MARTIN GONZALEZ, AND M. SILES MOLINA, The socle of a Leavitt path algebra, *J. Pure App. Alg.* **212** (2008), 500–509.
- [11] G. ARANDA PINO, D. MARTIN BARQUERO, C. MARTIN GONZALEZ, AND M. SILES MOLINA, Socle theory for Leavitt path algebras of arbitrary graphs, *Rev. Mat. Iberoamericana*, to appear. ArXiv: 0802.1198v1
- [12] G. ARANDA PINO, E. PARDO, AND M. SILES MOLINA, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305**(2) (2006), 912–936.
- [13] E. ARTIN, C.J. NESBITT AND R.M. THRALL, Rings with Minimum condition, University of Michigan Press, Ann Arbor, 1955.

- [14] G. BACCELLA, Semiartinian V-rings and semiartinian von Neumann regular rings, *J. Algebra*, **173** (1995), 587–612.
- [15] V. CAMILLO AND K. FULLER, On Loewy length of rings, *Pacific J. Math.* **53** (1974), 347–354.
- [16] J. DAUNS, Chains of modules with completely decomposable quotients, *Pacific J. Math.* **17** (1966), 235–242.
- [17] L.FUCHS, Torsion preradicals and ascending Loewy series of modules, *J. reine angew. Math.*, **239/240** (1970), 169–179.
- [18] N. JACOBSON, Structure of rings. Amer. Math. Soc. Colloquium Publ., vol. 37 (1968).
- [19] W. KRULL, Zur Theorie der allgemeinen Zahlringe, *Math. Ann.* **99** (1928), 51–70.
- [20] W. KRULL, Allgemeine Bewertungstheorie, *J. Reine Angew. Math.* **167** (1931), 160–196.
- [21] I. RAEBURN, Graph Algebras. CBMS Regional conference series in Math., vol. 103, Amer. Math. Soc., Providence (2005).
- [22] M. TOMFORDE, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* **318** (2007), 270–299.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, COLORADO SPRINGS CO 80933 U.S.A.

E-mail address: `abrams@math.uccs.edu`

E-mail address: `krangasw@uccs.edu`

DEPARTAMENTO DE ALGEBRA, GEOMETRÍA, Y TOPOLOGÍA, UNIVERSIDAD DE MÁLAGA, 29071 MÁLAGA, SPAIN.

E-mail address: `msilesma@uma.es`